# Antiproximinal Norms in Banach Spaces[1]

## J. M. Borwein

*Centre for Experimental and Constructive Mathematics, Simon Fraser University,
Burnaby, British Columbia, V5A 1S6, Canada*
E-mail: jborwein@cecm.sfu.ca

## M. Jiménez-Sevilla

*Dpto. de Análisis Matemático, Facultad de Ciencias Matemáticas,
Universidad Complutense de Madrid, Madrid, 28040, Spain*
E-mail: marjim@sunam1.mat.ucm.es

and

## J. P. Moreno

*Dpto. de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid,
Madrid, 28049, Spain*
E-mail: josepedro.moreno@uam.es

We prove that every Banach space containing a complemented copy of $c_0$ has an antiproximinal body for a suitable norm. If, in addition, the space is separable, there is a pair of antiproximinal norms. In particular, in a separable polyhedral space $X$, the set of all (equivalent) norms on $X$ having an isomorphic antiproximinal norm is dense. In contrast, it is shown that there are no antiproximinal norms in Banach spaces with the convex point of continuity property (CPCP). Other questions related to the existence of antiproximinal bodies are also discussed. © 2002 Elsevier Science (USA)

## 1. INTRODUCTION

The theory of approximation to convex sets plays a central role in the study of many geometrical problems in normed spaces. In this context, the

concept of *antiproximinal body* was introduced by Klee in the middle of the sixties [20]. A set $C$ in a Banach space $(X, \|\cdot\|)$ is said to be antiproximinal if no point in $X \setminus C$ has a nearest point in $C$. When $C = B_{|\cdot|}$, the unit ball of an equivalent norm $|\cdot|$ on $X$, $B_{|\cdot|}$ is antiproximinal if and only if $B_{\|\cdot\|}$ is, in its turn, antiproximinal in $(X, |\cdot|)$. According to [11] we say in this case that $\|\cdot\|$ and $|\cdot|$ are *antiproximinal norms* or, simply, *companion norms*. When, in addition, there is an isomorphism $T$ of $X$ onto itself such that $T(B_{\|\cdot\|}) = B_{|\cdot|}$ we say that $\|\cdot\|$ and $|\cdot|$ are *isomorphic companion norms*.

Edelstein and Thompson [12] proved that the usual norm in $c_0$ has an isomorphic companion norm. Later, Cobzas proved the same result for the usual norm in $c$, in any Banach space of continuous functions isomorphic to $c$ and, finally, in $c(X)$, the space of all $X$-valued convergent sequences (see [4–7]). The existence of companion norms in a more general class of spaces of continuous functions was established by Fonf [14] (e.g., for $C(K)$, where $K$ is the n-dimensional cube, the Cantor set or a metric compact containing the Hilbert cube). Later, Balaganskii [1] proved that for any $C(Q)$, where $Q$ is a topological space, the supremum norm has a closed, convex and bounded antiproximinal body. The study on the existence of ray-bounded (but non-necessarily bounded) antiproximinal bodies was considered in [24] by Phelps.

The first section of this paper is devoted to finding a class of Banach spaces admitting an antiproximinal body (a closed bounded convex set with non-empty interior). Regarding the preceding results, if the reader guesses that the obvious candidates are spaces containing $c_0$, he or she is guessing right. We prove that every Banach space with a complemented copy of $c_0$ has an antiproximinal body for a suitable norm. When, in addition, the space is separable, the existence of a pair of isomorphic companion norms is ensured. In particular, one of these norms is the supremum norm for an appropriate decomposition of the space as a sum $Y \oplus c_0$. This is not true in general, as G. Godefroy observed to us. Indeed, if $Y$ is a nonseparable reflexive Banach space, the sup norm on $Y \oplus c_0$ has no isomorphic companion. Further results in this section concern the density of norms admitting an isomorphic companion in separable polyhedral spaces. Fonf [15] considered the notion of a pair of *strongly antiproximinal norms*. A pair of strongly antiproximinal norms is not necessarily a pair of isomorphic companion norms and the reverse implication does not hold either. Fonf characterized the class of separable Banach spaces having a pair of strongly antiproximinal norms by those containing $c_0$.

Even complemented subspaces of a Banach space with a pair of isomorphic companion norms need not admit antiproximinal bodies. However, in Section 3 we show the following hereditary property: if $\|\cdot\|$ and $|\cdot|$ are companion norms in $X$, for every subspace $Y \subset X$ there is a subspace $Y \subset Z \subset X$ (with the same density of $Y$) such that $\|\cdot\|$ and $|\cdot|$ are

companion norms in $Z$. This hereditary property is similar to the one obtained in [3] for the Mazur intersection property in Asplund spaces and in [23] for property $\alpha$.

In spaces with the Radon Nikodym property (RNP) there are no antiproximinal bodies. This is due to the fact that the set of support functionals of every closed convex and bounded set is residual [2]. Indeed, a body is antiproximinal if and only if it has no common support functional with the unit ball. Spaces with the RNP are (strictly) contained in the class of Banach spaces with the *convex point of continuity property* (CPCP) [18]. Our aim in Section 4 is to show that spaces with the CPCP have no antiproximinal bodies either, despite of the fact that in these spaces the set of support functionals of a convex body need not be generic. In the last section we discuss some problems related to locally uniformly rotund norms and lattice norms. We finish that section with some Baire category considerations.

The next results are well known and they will be used without explicit mention in what follows. Two equivalent norms $\|\cdot\|$ and $|\cdot|$ are companions if and only if the sum of the unit balls $B_{\|\cdot\|} + B_{|\cdot|}$ is an open set. Recall that, if $X^*$ is the dual space of $X$ and $C \subset X$ is a nonvoid subset of $X$, a functional $f \in X^*$ is said to *support* $C$ if there exists $x \in C$ such that $f(x) = \sup_{y \in C} f(y)$ or $f(x) = \inf_{y \in C} f(y)$. Denote by supp $C$ the set of all functionals supporting $C$. Notice that if $C = B_{\|\cdot\|}$ the set supp $C$ consist of the norm attaining functionals on $(X, \|\cdot\|)$, which is usually denoted by $NA_{\|\cdot\|}$. The norms $\|\cdot\|$ and $|\cdot|$ are companions if and only if $NA_{\|\cdot\|} \cap NA_{|\cdot|} = \{0\}$. Analogously, the norm $\|\cdot\|$ and the body $C$ are companion if and only if $NA_{\|\cdot\|} \cap \operatorname{supp} C = \{0\}$. Assume, further, that $T$ is an isomorphism on $X$ such that $T(B_{|\cdot|}) = B_{\|\cdot\|}$ and denote by $T^*$ the adjoint operator $(T^*(f) = f \circ T)$. Since $T^*(NA_{\|\cdot\|}) = NA_{|\cdot|}$, the norms $\|\cdot\|$ and $|\cdot|$ are isomorphic companions if $T^*(NA_{\|\cdot\|}) \cap NA_{\|\cdot\|} = \{0\}$.

## 2. ANTIPROXIMINAL NORMS ON BANACH SPACES CONTAINING $c_0$

Given a family of non-trivial Banach spaces $\{X_i\}$, $i \in I$, with $I$ infinite, it is usual to denote by $(\sum_i X_i)_{c_0}$ the space of all functions $x: I \to \bigcup_i X_i$ such that $x(i) = x_i \in X_i$, $i \in I$, and the set $\{i \in I : \|x_i\| > \varepsilon\}$ is finite, for all $\varepsilon > 0$. The usual norm on $(\sum_i X_i)_{c_0}$ is given by $\|x\|_\infty = \sup \{\|x_i\|: i \in I\} = \max \{\|x(i)\|: i \in I\}$. Cobzas gave a proof in [6] of the existence of an isomorphic companion norm to the supremum norm $\|\cdot\|_\infty$ in $(\sum_i X_i)_{c_0}$, whenever $X_i = X$, for every $i \in I$. It is an open problem whether there is a pair of antiproximinal norms on any arbitrary sum $(\sum_i X_i)_{c_0}$. We shall prove in this section the existence of a pair of isomorphic companion norms on every separable Banach space $X$ which contains a copy of $c_0$. As

a corollary, we shall deduce the existence on $(\sum_n X_n)_{c_0}$ of an isomorphic companion norm for the supremum norm $\|\cdot\|_\infty$, whenever $X_n$ is separable for every $n \in \mathbb{N}$.

PROPOSITION 2.1.  *Every separable Banach space $Z$ containing $c_0$ admits a pair of isomorphic companion norms. In particular, if $(X_n)_n$ is a family of (non-trivial) separable Banach spaces, the supremum norm $\|\cdot\|_\infty$ in $(\sum_n X_n)_{c_0}$ admits an isomorphic companion norm.*

*Proof.*  Since, by Sobczyk's Theorem [21], $c_0$ is complemented in $Z$, we may assume $Z = X \oplus c_0$, where $X$ is a Banach space. Denote by $\|\cdot\|$ the norm on $X$ and by $|\cdot|$ the usual norm on $c_0$. Consider on $X \oplus c_0$ the supremum norm $\|(x, t)\|_\infty = \max\{\|x\|, |t|\}$, $(x \in X, t \in c_0)$, which is equivalent to the original norm given on $Z$. Then $\|\cdot\|_\infty^* = \|\cdot\|_1 : \|(x^*, t^*)\|_1 = \|x^*\|^* + |t^*|^*$, thus implying that $(x^*, t^*) \in NA_{\|\cdot\|_\infty}$ if and only if $x^* \in NA_{\|\cdot\|}$ and $t^* \in NA_{|\cdot|}$. In other words, $(x^*, t^*) \in NA_{\|\cdot\|_\infty}$ if and only if $x^* \in NA_{\|\cdot\|}$ and $t^*$ is a finite linear combination of $\{e_n^*\}$, the usual basis of $\ell_1$. We need to find an isomorphism $T: Z \to Z$ so that $T^*: X^* \oplus \ell_1 \to X^* \oplus \ell_1$ satisfies $T^*(NA_{\|\cdot\|_\infty}) \cap NA_{\|\cdot\|_\infty} = \{0\}$. Let $\{x_k\}$ be a countable dense set in $X$ and $0 < \varepsilon < 1$. For any element $(x^*, 0) \in X^* \oplus \ell_1$ define

$$T^*(x^*, 0) = x^* + \varepsilon \sum_k x^*(x_k)\, 2^{-2k} e_{2k+1}^*$$

and for the element $(0, e_n^*)$ set

$$T^*(0, e_n^*) = e_n^* + \varepsilon \sum_k 2^{-2k} e_{2^n(2k+1)}^*.$$

Since $T^*$ acting on $\ell_1$ is the same isomorphism as the one defined in [12], it follows that $T^*$ is an isomorphism from $X^* \oplus \ell_1$ onto itself.

Analogously, to see that $T^*$ is *weak\*–weak\** continuous (that is to say, $T^*$ is indeed the adjoint of an isomorphism $T$ on $Z$), it is enough to check that $\{T^*(x_\alpha^*)\}$ is weak\* convergent to $T^*(x^*)$ whenever $\{x_\alpha^*\}$ is weak\*-convergent to $x^*$ and $\{x_\alpha^*\}, x^* \subset X^*$. Indeed, for fixed $(x, t) \in X \oplus c_0$,

$$\lim_\alpha (T^*x_\alpha^* - T^*x^*)(x, t) = \lim_\alpha (x_\alpha^* - x^*)(x) + \varepsilon \sum_k (x_\alpha^* - x^*)(x_k)\, 2^{-2k} e_{2k+1}^*(t)$$

$$= \lim_\alpha (x_\alpha^* - x^*)(x) + \varepsilon \sum_k (x_\alpha^* - x^*)(2^{-2k} t_{2k+1} x_k)$$

$$= \lim_\alpha (x_\alpha^* - x^*)\left(x - \varepsilon \sum_k 2^{-2k} t_{2k+1} x_k\right) = 0.$$

To prove that $T^*(NA_{\|\cdot\|_\infty}) \cap NA_{\|\cdot\|_\infty} = \{0\}$, it is enough to observe that

$$T^*(x^* + t^*) = x^* + \text{an infinite linear combination of } \{e_n^*\},$$

whenever $x^* + t^* \neq 0$ and $t^*$ is a finite linear combination of $\{e_n^*\}$.

Notice that the same argument provides a companion norm to any given norm $\|\|\cdot\|\|$ on $Z$ which satisfies the following condition:

(*)  *There is an isomorphic embedding $i: c_0 \to Z$ so that if $f \in NA_{\|\|\cdot\|\|}$ then $i^*(f) \equiv \pi(f)$ is a finite linear combination of $(e_n^*)$, the usual basis of $\ell_1$.*

If this is the case, and we denote $i(c_0) \equiv c_0$, by Sobczyk's Theorem [21], $Z$ can be decomposed as $Z = X \oplus c_0$, for some Banach space $X$. Then, the result follows by considering the given isomorphism $T$ on $Z$. This is the case for the supremum norm $\|\cdot\|_\infty$ on $Z = (\sum_n X_n)_{c_0}$. Indeed, if an element $f \in Z^*$ attains its norm, then $f$ is a finite linear combination of the form $\sum_{i=1}^n \lambda_i f_i$, where $f_i \in X_i^*$ and $\lambda_i \in \mathbb{R}$. Select $y_n \in X_n$, $\|y_n\| = 1$, and consider the inclusion $i: c_0 \to (\sum_n X_n)_{c_0}$, defined by $i((x_n)_n) = \sum_n x_n y_n$, where $(x_n)_n \in c_0$ (which is an isometry of $c_0$ onto its image). Then, $i^*(f) \equiv \pi(f)$ is a finite linear combination of $(e_n^*)$, whenever $f \in NA_{\|\cdot\|_\infty}$. This finishes the proof of Proposition 2.1.  ∎

Denote by $N(Z)$ the set of all equivalent norms on $Z$ and consider in $N(Z)$ the uniform metric: $d(\|\cdot\|, |\cdot|) = \text{dist}(B_{\|\cdot\|}, B_{|\cdot|})$ i.e., the *Hausdorff distance* between their unit balls. It is well known that $(N(Z), d)$ is a Baire space. Note, in the above proof, $\|Id - T\| = \|Id - T^*\| \leqslant \varepsilon$, and this implies that, in the Hausdorff metric, we can approximate the supremum norm $\|\cdot\|_\infty$ (considered in $X \oplus c_0$ or in $(\sum_n X_n)_{c_0}$) arbitrarily well by a companion norm. Also, as we indicated in the above proof, the supremum norm $\|\cdot\|_\infty$ may be replaced by any other norm $\|\|\cdot\|\|$ on $Z$ satisfying (*).

It is natural to ask if the set of such norms is dense in $N(Z)$ for any separable Banach space $Z$ which contains $c_0$. We can ensure that this question has a positive answer when, in addition, $Z$ is *polyhedral* as we shall see in the next result. Recall that a Banach space $Z$ is polyhedral [13] if the intersection of any finite dimensional subspace of $Z$ with the closed unit ball is a polyhedron.

PROPOSITION 2.2.  *In a separable polyhedral Banach space $Z$, the set of norms admitting an isomorphic companion norm is dense in $N(Z)$.*

*Proof.*  By a theorem of Fonf [13], $c_0$ embeds into $Z$ and then $Z = X \oplus c_0$, for some Banach space $X$. In addition, every equivalent norm on $Z$ may be approximated in $N(Z)$ by a polyhedral norm $\|\cdot\|$ [10]. Moreover, we shall prove that every polyhedral norm $\|\cdot\|$ may be approximated by another polyhedral norm $|\cdot|$ which satisfies that, for every $f \in NA_{|\cdot|}$,

$\pi(f) \in \text{span}(e_n)$, where $\pi$ denotes the adjoint of the inclusion $i: c_0 \to Z$, $(e_n)$ are the elements of the canonical basis of $\ell_1$ and $\text{span}(e_n)$ denotes the finite linear combinations of $(e_n)$. In order to do that, let us consider a polyhedral norm $\|\cdot\|$, and denote by $(f_n)$ its countable boundary [13] (that is to say a countable collection with $\|f_n\| = 1$ and for every $x \in S_{\|\cdot\|}$ there exists $f_{n_0}$ so that $f_{n_0}(x) = 1$). Then, we fix $\varepsilon > 0$ and a decreasing sequence of real numbers $(\varepsilon_i) \searrow 0$ ($\varepsilon_1 \leq \varepsilon$). Select, for every $n \in \mathbb{N}$, an element $g_n \in Z^*$ so that $\pi(g_n) \in \text{span}(e_n)$ and $(1+\varepsilon_n)(1-\|f_n - g_n\|) > 1$. Consider the equivalent norm

$$|x| = \sup_n |(1+\varepsilon_n)\, g_n(x)|, \qquad x \in Z.$$

For every $x \in S_{\|\cdot\|}$ there is $n_0$ so that $f_{n_0}(x) = 1$. Thus,

$$(1+\varepsilon_{n_0})\, g_{n_0}(x) \geq (1+\varepsilon_{n_0})(1 - \|f_{n_0} - g_{n_0}\|) > 1 \tag{2.1}$$

and

$$\limsup_n |(1+\varepsilon_n)\, g_n(x)| = \limsup_n |(1+\varepsilon_n)\, f_n(x)| \leq 1. \tag{2.2}$$

Equations (2.1) and (2.2) imply that the norm $|\cdot|$ is polyhedral, the family $\{\pm(1+\varepsilon_n)\, g_n\}$ is a boundary and every $f \in NA_{|\cdot|}$ has the property that $\pi(f)$ is a finite linear combination of $(g_n)_n$ ([10, Theorem A]) and therefore $\pi(f)$ is a finite linear combination of $(e_n)$.

Finally, the norm $|\cdot|$ has an isomorphic companion norm. Indeed, If $T: X \oplus c_0 \to X \oplus c_0$ is the isomorphism considered in Proposition 2.1, then $B_{|\cdot|}$ and $T^{-1}(B_{|\cdot|})$ are the closed unit balls of a pair of companion norms on $Z$. ∎

In the next result we show that, in a sense, Proposition 2.1 is sharp. Indeed, we shall show that the assumption of separability cannot be removed.

PROPOSITION 2.3. *If $X$ is a non-separable and reflexive Banach space, then the supremum norm in the space $Z = X \oplus c_0$ does not have an isomorphic companion norm.*

*Proof.* Assume that there is an isomorphism $T: Z \to Z$, giving the companion norm for the sup norm. Then, if

$$T(x, y) = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

for every $x \in X$ and $y \in c_0$, consider $T_2: c_0 \to X$. Since $T_2$ does not have dense image, $T_2^*$ is not injective, i.e. $H = \ker T_2^* \neq \{0\}$. Notice that the projection $\pi_1: X \oplus c_0 \to X$ has norm 1, so $f \in X^*$ attains its norm in $X$ if and only if $f$ attains its norm in $Z$. Then, any non-zero $f \in H \subset X^*$, attains its norm in $Z$ but $T^*(f) = T_1^*(f) \in X^*$ does not attain its norm in $Z$, which is a contradiction because of the reflexivity of $X$. ∎

*Remark* 2.4. In particular, the sup norm in $\ell_2(c) \oplus c_0$ does not have any isomorphic companion norm. Let us notice that Proposition 2.3 remains true whenever the norm considered in $Z$ is such that the canonical projection $\pi: Z \to X$ has unit norm.

We finish this section with a positive result on the existence of an antiproximinal body in spaces containing a complemented copy of $c_0$, when the space is endowed with the sup norm associated with a suitable decomposition of the space.

PROPOSITION 2.5. *Let $X$ be an arbitrary Banach space and consider $c_0$ with its usual norm. Then the supremum norm in the space $Z = X \oplus c_0$ has an antiproximinal body*.

*Proof.* First, let us recall that if we consider the sup norm in $Z$, every norm attaining functional in $Z^*$ is of the form $x^* + r^*$, where $x^* \in X^*$ and $r^* \in \ell_1$ is a finite linear combination of the elements of the canonical basis $(e_n)$. Let us denote by $B_{\|\cdot\|}$ and $B_{\|\cdot\|^*}$ the closed unit ball in $Z$ and the dual closed unit ball in $Z^*$, respectively. We shall prove that, as in [1], the polar of an appropriate translate of $B_{\|\cdot\|^*}$ is an antiproximinal body to $B_{\|\cdot\|}$. Take $u = \sum_n \frac{1}{2^{n+1}} e_n \in \ell_1$ and define $C = B_{\|\cdot\|^*} + u$. Then C is a weak* closed, convex and bounded subset of $Z^*$ with $0 \in \text{int } C$. The polar set $D$ of $C$ in $Z$, that is to say, $D = \{z \in Z : f(z) \leq 1 \text{ for every } f \in C\}$, is a bounded, closed and convex subset of $Z$ with $0 \in \text{int } C$. Let us prove that $D$ and $B_{\|\cdot\|}$ are an antiproximinal pair. Assume that $f \in Z^*$ supports $D$ at a point $x$. We even may assume that $1 = f(x) = \sup_D f$ (the case $f(x) = \inf_D f = -1$ is similar). Then, $f \in C$ and $x$ supports $C$ at $f$. Since $C = B_{\|\cdot\|^*} + u$, we have that $x$ supports $B_{\|\cdot\|^*}$ at $f - u$. So $i^*(f - u)$ is a finite linear combination of $(e_n)$, where $i^*$ is the adjoint of the canonical inclusion $i: c_0 \to Z$. Finally this implies that $i^*(f)$ is an infinite linear combination of the basis $(e_n)$ and therefore $f$ is not a norm attaining functional for the supremum norm in $Z$. ∎

## 3. AN HEREDITARY PROPERTY

As a consequence of the previous results, even complemented subspaces of a Banach space with companion norms need not admit such a pair of

norms. This section is devoted to showing that, nonetheless, there is in this context an hereditary property similar to the one obtained in [3] for the Mazur intersection property in Asplund spaces and in [23] for property $\alpha$. Denote by dens $X$ the *density character* of $X$, i.e. the smallest cardinal number of a dense subset in $X$.

PROPOSITION 3.1.    *Let $\|\cdot\|$ and $|\cdot|$ be two companion norms on $X$. For every subspace $Y \subset X$, there exists a subspace $Z \subset X$ with $Y \subset Z$ and* dens $Y = $ dens $Z$ *such that $\|\cdot\|$ and $|\cdot|$ are companion norms on $Z$.*

*Proof.*    Let $\{y_\alpha^1\}_{\alpha \in \Gamma_1}$ be a dense subset in $Y \setminus B_{|\cdot|}$. For each $\alpha \in \Gamma_1$ there is a minimizing sequence $\{x_{\alpha k}^1\}_{k=1}^\infty \subset B_{|\cdot|}$ such that $\text{dist}_{\|\cdot\|}(y_\alpha^1, B_{|\cdot|}) = \lim_k \|y_\alpha^1 - x_{\alpha k}^1\|$. Define $Y = Y_1$ and

$$Y_2 = \overline{\text{span}} \left\{ \bigcup_{\alpha \in \Gamma_1} \{x_{\alpha k}^1\}_{k=1}^\infty \cup Y_1 \right\}.$$

Assume that $Y_n$ has been defined. Again, if $\{y_\alpha^n\}_{\alpha \in \Gamma_n}$ is a dense subset of $Y_n \setminus B_{|\cdot|}$ with card $\Gamma_n = $ dens $Y_n$, for each $\alpha \in \Gamma_n$ there is a minimizing sequence $\{x_{\alpha k}^n\}_{k=1}^\infty \subset B_{|\cdot|}$ satisfying $\lim_k \|y_\alpha^n - x_{\alpha k}^n\| = \text{dist}_{\|\cdot\|}(y_\alpha^n, B_{|\cdot|})$. Define

$$Y_{n+1} = \overline{\text{span}} \left\{ \bigcup_{\alpha \in \Gamma_n} \{x_{\alpha k}^n\}_{k=1}^\infty \cup Y_n \right\}.$$

So far we have a sequence of subspaces $\{Y_n\}_{n=1}^\infty \subset X$ satisfying $Y_n \subset Y_{n+1}$ and dens $Y_n = $ dens $Y_{n+1}$. Set $Z = \overline{\text{span}}\,(\bigcup_n Y_n)$. Clearly, $Y \subset Z$ and dens $Z = $ dens $Y$. In order to finish the proof we just need to verify that no point in $Z \setminus B_{|\cdot|}$ has a best approximation for the norm $\|\cdot\|$ in $B_{|\cdot|}$. To this end, it is enough to observe that the two distance functions $d_1(x) = \text{dist}_{\|\cdot\|}(x, B_{|\cdot|}(X))$ and $d_2(x) = \text{dist}_{\|\cdot\|}(x, B_{|\cdot|}(Z))$ agree on the set of points $D = \{y_\alpha^n, \alpha \in \Gamma_n, n \in \mathbb{N}\}$. It is then readily verified that $D$ is dense in $Z$ and, therefore, by continuity, $d_1$ and $d_2$ agree on $Z$.    ∎

## 4. THERE ARE NO ANTIPROXIMINAL BODIES IN SPACES WITH THE CPCP

It is known that a Banach space has the *Radon Nikodym property* (RNP) if and only if, for every equivalent norm, the set of functionals supporting the unit ball is residual [22]. Consequently, spaces with the RNP do not admit companion norms. A natural question is to ask whether this result can be generalized to spaces with the Convex Point of Continuity Property, which is known to be strictly weaker than the RNP [18]. Recall that a *point of continuity* of $C \subset X$ is a point at which the relative norm topology

and the weak topology coincide on $C$. A Banach space has the *convex point of continuity property* (CPCP, for short) if every bounded closed convex subset of $X$ has a point of continuity.

We proceed to show that, in a Banach space $X$ with the CPCP, every countable family of closed, convex and bounded sets has a dense set of common support functionals, despite the fact that the set of support functionals of a closed, convex, bounded set need not be generic in $X^*$.


PROPOSITION 4.1.   *Consider two closed convex and bounded sets $A$, $B$ in $X$ and let $C$ be the closure of $A+B$. Every point of continuity of $C$ is the sum of a point of continuity of $A$ and a point of continuity of $B$. Consequently, every functional supporting $C$ at a point of continuity simultaneously supports both $A$ and $B$.*

*Proof.*   Let $x$ be a point of continuity of $C$ and let $\bar{C}^{w^*}$ be the weak$^*$ closure of $C$ in the bidual $X^{**}$. Then $x$ is a point of weak$^*$-norm continuity of $\bar{C}^{w^*}$. Now, since $\bar{C}^{w^*} = \bar{A}^{w^*} + \bar{B}^{w^*}$, there are $F_1 \in \bar{A}^{w^*}$ and $F_2 \in \bar{B}^{w^*}$ such that $x = F_1 + F_2$. It can be readily verified that $F_1$ and $F_2$ are weak$^*$-norm points of continuity of $\bar{A}^{w^*}$ and $\bar{B}^{w^*}$ respectively so, actually, $F_1 \in A$ and $F_2 \in B$. The second part of the proposition is immediate, since every functional supporting $C$ at $x$ supports also $A$ at $F_1$ and $B$ at $F_2$.   ∎

Notice that one direct consequence of the preceding result is that whenever $A$ and $B$ are two closed convex sets in a Banach space $X$ with the CPCP, the set $A+B$ is never open. Indeed, $A+B$ contains all the points of continuity of its closure.


COROLLARY 4.2.   *If a Banach space has the CPCP, then it does not admit a pair of companion norms.*

Recall that in a Banach space $X$ with the CPCP, for every closed, convex and bounded set $C \subset X$, the set of support functionals at points of continuity of $C$ is dense in $X^*$ [9, Proposition 7]. Thence, we may establish the following result.


PROPOSITION 4.3.   *Let $\{C_n\}$ be a family of closed, convex and bounded sets in a Banach space $X$ with the CPCP. There is a dense set $F \subset X^*$ so that each $f \in F$ supports $C_n$ for each $n \in \mathbb{N}$.*

*Proof.*   We may assume that $C_n \subset B_{\|\cdot\|}$ for every $n \in \mathbb{N}$. Otherwise it is enough to consider the sets $C_n' = \lambda_n C_n$ with $\lambda_n^{-1} = \sup \{\|x\| : x \in C_n\}$ since $C_n$ and $C_n'$ have the same supporting functionals. Define $C$ as the closure of

$\sum_n 2^{-n}C_n$ and let $f \in X^*$ a support functional of $C$ at a point $x$ of continuity of $C$. Then $x$ is a point of weak*-norm continuity of

$$\bar{C}^{w^*} = \overline{2^{-1}C_1}^{w^*} + \overline{\sum_{n>1} 2^{-n}C_n}^{w^*}.$$

Therefore, $x = 2^{-1}x_1 + F_2$, where $x_1$ is a point of continuity of $C_1$ and $F_2$ is a point of weak*-norm continuity of $\overline{\sum_{n>1} 2^{-n}C_n}^{w^*}$. Now proceeding inductively, it is possible to find a sequence $\{x_n\}$ so that $x_n$ is point of continuity of $C_n$ and $x = \sum_n 2^{-n}x_n$. Just as in Proposition 4.1, it is clear that $f$ is a supporting functional of $C_n$ at $x_n$, for each $n \in \mathbb{N}$.  ∎

Recall that, given a closed set $C$ in a Banach space $(X, \|\cdot\|)$ and $x \in C$, we say that $x$ is a *nearest point* if there is $y \notin C$ such that $\|x-y\| = \text{dist}(y, C)$. The point $x \in C$ is a nearest point of $C$ if and only if there is $f \in NA_{\|\cdot\|}$ supporting $C$ at $x$. We finish this section with a result on the density of nearest points.

PROPOSITION 4.4. *Let $X$ be a Banach space with the CPCP. Every closed, bounded, and convex set $C \subset X$ has a weakly dense set of nearest points.*

*Proof.* Consider a non–empty relatively weakly open set $W = \{x \in C : f_i(x) > \lambda_i, i = 1, ..., n\}$ in $C$, for some $n \in \mathbb{N}$ and $\|f_i\|^* = 1$. Let us take $V = \{x \in C : f_i(x) > \lambda_i', i = 1, ..., n\}$ where we choose $\lambda_i' > \lambda_i$ appropriately so that $V$ is non–empty. Denote by $C_\varepsilon$ the closure of the set $C + \varepsilon B_{\|\cdot\|}$, where $0 < \varepsilon < \min_i (\lambda_i' - \lambda_i)$. By [19, Lemma 3.15], the set of points of continuity of $C_\varepsilon$ is weakly dense in $C_\varepsilon$. Therefore there is a point of continuity $c$ of $C_\varepsilon$ in $\tilde{V} = \{x \in C_\varepsilon : f_i(x) \geqslant \lambda_i', i = 1, ..., n\}$. Then $c$ can be expressed as $c = c_1 + \varepsilon c_2$ where $c_1$ and $c_2$ are points of continuity of $C$ and $B_{\|\cdot\|}$, respectively. Then, there is $f \in X^* \setminus \{0\}$ supporting $C$ and $B_{\|\cdot\|}$ at $c_1$ and $c_2$, respectively. Finally, $c_1 = c - \varepsilon c_2 \in W$ is a nearest point of $C$.  ∎

## 5. COMMENTS AND CONCLUDING REMARKS

### 1. *The Case of Locally Uniformly Rotund Norms*

Edelstein and Thompson gave an example in [12] of a rotund norm admitting a companion and asked whether it was possible to find a locally uniformly rotund norm with the same property. Recall that a norm $\|\cdot\|$ is said to be *locally uniformly rotund* if $\lim_n \|x_n - x\| = 0$ whenever $\|x_n\| = \|x\| = 1$ and $\lim_n \|x_n + x\| = 2$. In [8], the first example of a norm admitting a locally uniformly rotund companion was exhibited. Later, in [11], it was

proved that *Day's norm* on $c_0$, which is locally uniformly rotund, itself admits a companion. That said, locally uniformly rotund norms cannot admit isomorphic companions.

We shall show that a norm $\|\cdot\|$ with a residual set of support functionals does not have any isomorphic companion norm. Indeed, if $T^*$ is an isomorphism from $X^*$ into itself and $NA_{\|\cdot\|}$ is residual, it is clear that $T^*(NA_{\|\cdot\|})$ is residual, and hence $NA_{\|\cdot\|} \cap T^*(NA_{\|\cdot\|})$ is residual, too. On the other hand it is well known that $NA_{\|\cdot\|}$ is certainly residual whenever $\|\cdot\|$ is locally uniformly rotund. We give the proof of this fact for the sake of completeness. To this end, fix $x \in X$, $\|x\| = 1$ and $f \in X^*$ so that $f(x) = \|f\|^*$. Observe that $\lim_n \|x - x_n\| = 0$ whenever $\|x_n\| \leqslant 1$ and $\lim_n f(x_n) = 1$. Consider now, for every $n \in \mathbb{N}$, the set

$$X_n^* = \left\{ f \in X^* : \text{there is } 0 < \lambda < \|f\|^* \text{ with diam } \{y \in B_{\|\cdot\|} : f(y) > \lambda\} < \frac{1}{n} \right\}$$

which is open and, by the Bishop–Phelps theorem and the previous discussion, dense. Then $\bigcap_n X_n^* \subset NA_{\|\cdot\|}$ thus implying that $NA_{\|\cdot\|}$ is residual.

Relatedly, in locally uniformly rotund Banach spaces, every locally uniformly rotund body $C$ has a dense set of nearest points in $\partial C$ (see [17] for the definition of locally uniformly rotund body). This is a consequence of the fact that the set of common support functionals to $C$ and the unit ball is $G_\delta$ and dense and the norm topology in $\partial C$ is given by slices.

## 2. *The Case of Lattice Norms*

Recall that $(X, \|\cdot\|)$ is a Banach lattice when $X$ is a vector lattice and the norm $\|\cdot\|$ is complete and satisfies $\|x\| \leqslant \|y\|$ whenever $|x| \leqslant |y|$. Let $X$ be a Banach lattice satisfying $X = X_1 \oplus X_2$, where $X_1$ and $X_2$ (two nontrivial subspaces) are lattice orthogonal and $X_1$ does not possess any pair of companion norms (for instance, if $X_1$ is finite dimensional). Then, no pair of lattice norms on $X$ can be companions. Examples of spaces satisfying these properties are (i) sequential Banach lattices, (ii) $L_1(\mu)$ whenever $\mu$ is a measure with at least one atom, and (iii) $C(K)$ for any compact Hausdorff $K$ with at least one isolated point.

To prove the above statement first notice that, for any lattice norm $\|\cdot\|$ on $X$, the projection $\pi_1 : X \to X_1$ has norm 1 (considering in $X_1$ the restriction of the norm $\|\cdot\|$). Indeed, if $x \in X_1$ and $y \in X_2$, then by hypothesis $|x| \wedge |y| = 0$. This implies that $|x + y| = |x| + |y|$ and, consequently, $\|x\| \leqslant \|x + y\|$ (see [25]). Thus, a functional $f \in X_1^*$ attains its norm on $X_1$ if and only if $i(f)$, attains its norm on $X$, where $i : X_1^* \to X^*$ denotes the canonical inclusion. Therefore, if $f \neq 0$ is a common norm attaining functional in $X_1^*$ for a pair of lattice norms in $X_1$ (which are the restrictions of a pair of

lattice norms in $X$), then $i(f)$ is a common norm attaining functional in $X^*$ for the pair of lattice norms in $X$. Consequently, these norms are not companions.

## 3. *A Baire Category Remark*

If we fix a norm $\|\cdot\|$ on $X$, it is possible to prove that the set of companion norms to $\|\cdot\|$ is first Baire category in $N(X)$. Indeed, it is enough to show that for a fixed $f \in NA_{\|\cdot\|}$, there is a residual set of norms having $f$ as a support functional. This assertion can be proved by observing that the set of dual norms which are Fréchet differentiable at $f$ is residual. The proof of this fact is analogous to the one given in [16, 26]. Nevertheless, it is unknown whether the set of norms admitting a companion is first Baire category in $N(X)$.

## ACKNOWLEDGMENTS

## REFERENCES

1. V. S. Balaganskii, Antiproximal sets in spaces of continuous functions, *Math. Notes* **60**, No. 5 (1996), 485–494.

2. R. D. Bourgin, "Geometric Aspects of Convex Sets with the Radon–Nikodým Property," Lecture Notes in Math., Vol. 993, Amer. Math. Soc., Providence, 1983.

3. Dongjian Chen, Zhibao Hu, and Bor-Luh Lin, Ball intersection properties of Banach spaces, *Bull. Austral. Math. Soc.* **45** (1992), 333–342.

4. S. Cobzas, Convex antiproximal sets in the spaces $c_0$ and $c$, *Mat. Zametki* **17** (1975), 449–457. [In Russian]

5. S. Cobzas, Antiproximinal sets in Banach spaces of continuous functions, *Rev. Anal. Numér. Théor. Approx.* **5** (1976), 127–143.

6. S. Cobzas, Antiproximinal sets in Banach spaces of $c_0$-type, *Rev. Anal. Numér. Théor. Approx.* **7** (1978), 141–145.

7. S. Cobzas, Antiproximinal sets in the Banach space $c(X)$, *Comment. Math. Univ. Carolin.* **38**, No. 2 (1997), 247–253.

8. S. Cobzas, Multimifoarte neproximinale in $c_0$, *Rev. Anal. Numér. Théor. Approx.* **2** (1973), 137–141.

9. R. Deville, G. Godefroy, Hare, and V. Zizler, Differentiability of convex functions and the convex point of continuity property in Banach spaces, *Israel J. Math.* **59**, No. 2 (1987), 245–255.

10. R. Deville, P. Hajek, and V. Fonf, Analytic and polyhedral approximation of convex bodies in separable polyhedral Banach spaces, *Israel J. Math.* **105** (1998), 139–154.

11. M. Edelstein, Antiproximinal sets, *J. Approx. Theor.* **49** (1987), 252–255.
12. M. Edelstein and A. C. Thompson, Some results on nearest point and support properties of convex sets in $c_0$, *Pacific J. Math.* **40** (1972), 553–560.
13. V. P. Fonf, Some properties of polyhedral Banach spaces, *Funktsional. Anal. i Prilozhen.* **14** (1980), 89–90 [In Russian]; translation, *Functional Anal. Appl.* **14**, No. 4 (1980), 323–324.
14. V. P. Fonf, On antiproximinal sets in spaces of continuous functions on compacta, *Mat. Zametki* **33** (1983), 549–558 [In Russian]; translation, *Math. Notes* **33**, No. 3–4 (1983), 282–287.
15. V. P. Fonf, Strongly antiproximinal sets in Banach spaces, *Mat. Zametki* **47**, No. 2 (1990), 130–136; translation, *Math. Notes* **47** No. 1–2 (1990), 212–217.
16. P. G. Georgiev, Fréchet differentiability of convex functions in separable Banach spaces, *C. R. Acad. Bulgare Sci.* **43** (1990), 13–15.
17. P. G. Georgiev, A. S. Granero, M. Jiménez-Sevilla, and J. P. Moreno, Mazur intersection property and differentibility of convex functions, *J. London Math. Soc. (2)* **61** (2000), 531–542.
18. N. Ghoussoub, B. Maurey, and W. Schachermayer, A counterexample to a problem on points of continuity in Banach spaces, *Proc. Amer. Math. Soc.* **99** (1987), 278–282.
19. M. Jiménez-Sevilla and J. P. Moreno, A note on norm attaining functionals, *Proc. Amer. Math. Soc.* **126**, No. 7 (1998), 1989–1997.
20. V. Klee, Remarks on nearest points in normed linear spaces, *in* "Proc. Colloquium on Convexity, Copenhagen, 1965," pp. 168–176.
21. J. Lindenstrauss and L. Tzafriri, "Classical Banach Spaces. I. Sequences Spaces," Springer-Verlag, Berlin, 1979.
22. W. B. Moors, The relationship between Goldstine's Theorem and the convex point of continuity property, *J. Math. Anal. Appl.* **188** (1994), 819–832.
23. J. P. Moreno, Geometry of Banach spaces with property α, *J. Math. Anal. Appl.* **201** (1996), 600–608.
24. R. R. Phelps, Counterexamples concerning support theorems for convex sets in Hilbert spaces, *Canad. Math. Bull.* **31**, No. 1 (1988), 121–128.
25. H. H. Shaefer, "Banach Lattices and Positive Operators," Springer-Verlag, New York/ Berlin, 1974.
26. J. Vanderwerff, Fréchet differentiable norms on spaces of countable dimension, *Arch. Math.* **58** (1992), 471–476.